Besov Smoothness Priors for Wavelet Snakes

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ABSTRACT: This paper proposes a new model for contour deformations using wavelets. This model extends to contours the relationship between the wavelet decomposition of real functions and Besov spaces. As Besov spaces are smoothness spaces this allows to control the smoothness of the contour deformation extending previous wavelet representations. We also introduce a wavelet probabilistic model that induces a prior distribution for contour deformation. This allows to solve the fitting problem in bayesian terms. Computational results are presented that show applications of the smoothness model.

Keywords: Active contours, deformation modelling, wavelets, besov spaces, contour fitting.

1. INTRODUCTION

Deformable models [1] are object models suitable for representing non-rigid objects. A deformable model can be described by a vector of parameters that span a multidimensional space. Usually prior information about preferred deformations of the model can be obtained. This preference is then expressed in terms of a deformation energy that penalizes some deformations. The importance of this prior information is that it simplifies and increases the robustness of the solutions to some problems like fitting and tracking.

The deformation energy is coupled with a data mismatch criterion that measures the degree of discrepancy between the model and some measures extracted from an image. Model matching can then be formulated as an optimisation problem of a function that is defined in terms of both the deformation and the mismatch energy.

An important subset of deformable models is active contour models. Several parametric representations have been used to represent deformable contours. In this paper a new prior model for the fitting problem is presented. This model is based on a wavelet representation of shape [2] and Besov smoothness spaces. There are a number of salient features in wavelet transforms that make wavelet-domain contour processing attractive [3]:

- Locality: Each wavelet coefficient represents the signal content localised in spatial location and frequency.
- Multiresolution: The wavelet transform analyses the signal at a nested set of scales.
- Energy Compaction: A wavelet coefficient is large only if singularities are present within the support of the wavelet basis. Therefore we need to model only a small number of coefficients.
- Decorrelation: The wavelet transform of real world signals tend to be approximately decorrelated.

We will use the relation between wavelets and smoothness spaces to build wavelet contour representations with a desired degree of smoothness. In contrast with other active contour approaches like snakes [4] different levels of smoothness can be defined and smooth deformations can be enforced without the necessity to alter the balance between the uncertainty of the prior model for deformations and data extracted from the image. This wavelet representation is also set in probabilistic terms so that the realisations of the stochastic model are almost sure in a predefined smoothness space. We then show how this model can be employed to solve the fitting problem. This paper is divided in five parts: in Section 2 a wavelet based deformation model is presented and its relation with Besov spaces is established. In Section 3 a prior probabilistic distribution for contour deformation is introduced. In Section 4 we solve the fitting problem and show some computational results. Finally in Section 5 we present the conclusions of this paper.

2. WAVELET- BASED REPRESENTATIONS OF SHAPE

In this section wavelet shape representation is presented and its relation to Besov smoothness spaces is established. This will define a model for shape deformation that will be set in probabilistic terms and will be used to solve the fitting problem.

2.1 Wavelet Shape Representation

A wavelet basis uses translations and dilations of a scaling function ϕ and a wavelet function ψ. If a curve $\boldsymbol{r}(u)$ is closed and parameter u belongs to an interval I=[0,L] then we obtain the representation:

$$\boldsymbol{r}(u) = \boldsymbol{c}_{0,0}\phi_{0,0}(u) + \sum_{\substack{j\geq 0 \\ 0\leq l<2^j}} \boldsymbol{d}_{j,l}\psi_{j,l}(u)\,, \tag{1}$$

with the coefficients in the curve expansion defined as:

$$\boldsymbol{c}_{0,0} = \left(\left\langle x(u),\phi_{0,0}(u)\right\rangle,\left\langle y(u),\phi_{0,0}(u)\right\rangle\right)^{\mathrm{T}}, \quad \boldsymbol{d}_{j,l} = \left(\left\langle x(u),\psi_{j,l}(u)\right\rangle,\left\langle y(u),\psi_{j,l}(u)\right\rangle\right)^{\mathrm{T}}, \tag{2}$$

where: $\left\langle f, g\right\rangle = \frac{1}{L}\int_0^L f(u)\, g(u) du$.

Dilations and translations of scaling and wavelet functions are normalised so that: $\left\|\phi_{0,0}\right\|_{L_2(I)} = \left\|\psi_{j,l}\right\|_{L_2(I)} = 1$ and norm is defined as: $\left\| f\right\|^2_{L_2(I)} = \left\langle f, f\right\rangle$.

In practice, we will work with finite number of coefficients in the representation:

$$\boldsymbol{r}(u) = \boldsymbol{c}_{0,0}\phi_{0,0}(u) + \sum_{\substack{0\leq j<J \\ 0\leq l<2^j}} \boldsymbol{d}_{j,l}\psi_{j,l}(u)\,, \tag{3}$$

obtaining a curve with 2N degrees of freedom ($N = 2^J$).

Curve expansion can be concisely written in matrix form. First we express scaling and wavelet functions in a vector as:

$$\boldsymbol{G}_W(u) = (\phi_{0,0}(u) \quad \psi_{0,0}(u) \quad \ldots \quad \psi_{J-1,2^{J-1}-1}(u))^{\mathrm{T}}\,, \tag{4}$$

with scaling and wavelet coefficients in vectors $\boldsymbol{w}^x$ and $\boldsymbol{w}^y$ defined as:

$$\boldsymbol{w}^x = (c_{0,0;x} \quad d_{0,0;x} \quad \ldots \quad d_{J-1,2^{J-1}-1;x})^{\mathrm{T}}\,,\ \ \boldsymbol{w}^y = (c_{0,0;y} \quad d_{0,0;y} \quad \ldots \quad d_{J-1,2^{J-1}-1;y})^{\mathrm{T}}\,. \tag{5}$$

Then to describe curve $\boldsymbol{r}$(u) in matrix form we define:

$$\boldsymbol{F}_W(u) = \boldsymbol{I}_2 \otimes \boldsymbol{G}_W(u)^{\mathrm{T}} = \begin{pmatrix} \boldsymbol{G}_W(u)^{\mathrm{T}} & \boldsymbol{0}_N^{\mathrm{T}} \\ \boldsymbol{0}_N^{\mathrm{T}} & \boldsymbol{G}_W(u)^{\mathrm{T}} \end{pmatrix}, \tag{6}$$

where $\otimes$ denotes the Kronecker product and $\boldsymbol{0}_N$ a null vector of N components. Then we may write:

$$\boldsymbol{r}(u) = \boldsymbol{F}_W(u)\boldsymbol{w}\,, \qquad \boldsymbol{w}=(\boldsymbol{w}^{x\,\mathrm{T}}, \boldsymbol{w}^{y\,\mathrm{T}})^{\mathrm{T}}. \tag{7}$$

2.2 Wavelet based deformation modelling

In order to control the smoothness of deformations we will consider Besov spaces $B_q^\alpha(L_p(I))$, $0 < \alpha < \infty$, $0 < p \le \infty$, $0 < q \le \infty$. These spaces have, very roughly speaking, "α derivatives in $L_p(I)$". We propose to extend Besov spaces from real functions to contours defining a norm for curves equivalent to the "natural" norm in $B_q^\alpha(L_p(I)) \times B_q^\alpha(L_p(I))$ using:

Theorem 1

Let a curve $C \equiv \boldsymbol{r}(u) = (x(u), y(u))^T$ be decomposed on its wavelet representation then its "natural" Besov norm $\|\boldsymbol{r}\|_{B_q^\alpha(L_p(I))}$ is equivalent to the expression:

$$\|\boldsymbol{r}\|_{B_q^\alpha(L_p(I))} \equiv \sqrt{|c_{0,0;x}|^2 + |c_{0,0;y}|^2} + \left(\sum_{j\ge 0} \left(\sum_{k=0}^{2^j-1} 2^{\alpha jp} 2^{j(p/2-1)} \left(\sqrt{|d_{j,k;x}|^2 + |d_{j,k;y}|^2} \right)^p \right)^{q/p} \right)^{1/q} . \qquad \textbf{(8)}$$

Theorem 1 can be proved noting that the "natural" Besov norm $\|\boldsymbol{r}\|_{B_q^\alpha(L_p(I))} = \|x\|_{B_q^\alpha(L_p(I))} + \|y\|_{B_q^\alpha(L_p(I))}$ is equivalent to the expression above since two constants K, L can be found such that:

$$K\left(\|x\|_{B_q^\alpha(L_p(I))} + \|y\|_{B_q^\alpha(L_p(I))} \right) \le \sqrt{|c_{0,0;x}|^2 + |c_{0,0;y}|^2} + \left(\sum_{j\ge 0} \left(\sum_{k=0}^{2^j-1} 2^{\alpha jp} 2^{j(p/2-1)} \left(\sqrt{|d_{j,k;x}|^2 + |d_{j,k;y}|^2} \right)^p \right)^{q/p} \right)^{1/q} \le \qquad \textbf{(9)}$$

$$L\left(\|x\|_{B_q^\alpha(L_p(I))} + \|y\|_{B_q^\alpha(L_p(I))} \right) .$$

3. WAVELET PROBABILISTIC DEFORMATION MODELS IN BESOV SPACES

The simplest wavelet transform statistical models are obtained assuming that the coefficients are independent. Under the independence assumption, modelling reduces to simply specifying the marginal distribution of each wavelet coefficient.

The primary independent models employed to date are the generalised gaussian distribution GGD (related to Besov spaces) and the gaussian mixture distribution.

To extend the GGD we define the $\mathrm{GGSD}_\nu(\mathbf{0}_2,\sigma^2)$ distribution with mean $\mathbf{0}_2$, shape parameter ν and variance σ^2 as a bidimensional symmetric extension to the GGD. This leads the density function:

$$f(x,y) = \frac{\nu \eta_2(\nu)^2}{2\Gamma(2/\nu)} \frac{1}{\pi\sigma^2} \exp\left(-\left(\eta_2(\nu) \frac{\sqrt{x^2+y^2}}{\sigma} \right)^\nu \right), \quad \eta_2(\nu) = \sqrt{\frac{\Gamma(4/\nu)}{2\Gamma(2/\nu)}} , \qquad \textbf{(10)}$$

where $\sigma^2 = E(x^2) = E(y^2)$

It can be shown that the GGSD is related to Besov spaces using the following result that extends the theorem in [3]:

Theorem 2

Let a curve $\mathbf{C} \equiv \boldsymbol{r}(u) = (x(u), y(u))^T$ be decomposed on its wavelet representation and suppose that the vectors of its wavelet representation $\boldsymbol{d}_{j,k}$ are independently and identically distributed as: $\boldsymbol{d}_{j,k} = (d_{j,k;x}, d_{j,k;y})^T \sim \mathrm{GGSD}_p(\mathbf{0}_2, \sigma_j^2)$, $\sigma_j = 2^{-j\beta}\sigma_0$ with $\beta>0$ and $\sigma_0>0$. Then for $0<p$, $q<\infty$ the realisations of the model are almost surely in $B_q^\alpha(L_p(I)) \times B_q^\alpha(L_p(I))$ if and only if $\beta > \alpha + 1/2$.

Theorem 2 can be proved from the unidimensional version in [3].

To complete the definition of the model we have to specify the distribution for the coefficient associated with the scaling function $\boldsymbol{c}_{0,0.}$ This coefficient is associated with a translation of shape and we will assume that it is normally distributed and independent of the non-translation components $\boldsymbol{d}_{j,k}.$ This obviously does not change the pertenence of the curve to its Besov Space. Then:

$$\boldsymbol{c}_{0,0} = \begin{pmatrix} c_{0,0,x} \\ c_{0,0,y} \end{pmatrix} \sim \boldsymbol{N}(\boldsymbol{0}_2, \sigma_{TR}^2 \boldsymbol{I}_2) \ . \qquad \textbf{(11)}$$

Now using Theorem 2 if we define the prior density in wavelet space:

$$f_B(\boldsymbol{w}) \propto \exp\left(-\frac{1}{2\sigma_{TR}^2} \left(c_{0,0;x}{}^2 + c_{0,0;y}{}^2 \right) \right) \exp\left(\sum_{j \geq 0} \sum_{k=0}^{2^j - 1} \left(\frac{\eta_2(p)}{\gamma_{jk}} \sqrt{d_{j,k;x}{}^2 + d_{j,k;y}{}^2} \right)^p \right), \quad \gamma_{jk} = \sigma_{DEF} 2^{-\beta j} \qquad \textbf{(12)}$$

The realisations of the model are almost surely in $B_p^{\alpha}(L_p(I))$ if and only if $\beta > \alpha + 1/2$.

Covariance matrix for the probabilistic deformation model in (**10**) and (**11**) can be expressed in matrix form as:

$$\boldsymbol{\Sigma} = \boldsymbol{I}_2 \otimes \begin{pmatrix} \sigma_{TR}^2 & 0 & \ldots & 0 \\ 0 & \sigma_{DEF}^2 2^{-2\beta 0} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \sigma_{DEF}^2 2^{-2\beta(J-1)} \end{pmatrix} \qquad \textbf{(13)}$$

As shown before, parameter β is related to the smoothness of the deformation. It can be shown that parameters σ_{TR}^2 and σ_{DEF}^2 are related to the uncertainty of the contour. If a curve $\mathbf{C} \equiv \boldsymbol{r}(u) = (x(u), y(u))^{\mathrm{T}}$ is decomposed on its wavelet representation and its deformations are given by the wavelet probabilistic model shown in (**10**) and (**11**), then the mean square displacement $\bar{\rho}^2$ verifies:

$$\bar{\rho}^2 = \mathrm{Trace}(\boldsymbol{\Sigma}) \qquad \textbf{(14)}$$

This result, that is adapted from [5], allows the mean square displacement $\bar{\rho}^2$ to be decomposed in terms of the mean square displacement due to translation and non-translation $\bar{\rho}_{TR}^2$ and $\bar{\rho}_{DEF}^2$ as:

$$\bar{\rho}^2 = \bar{\rho}_{TR}^2 + \bar{\rho}_{DEF}^2, \quad \bar{\rho}_{TR}^2 = 2\sigma_{TR}^2, \quad \bar{\rho}_{DEF}^2 = 2\sigma_{DEF}^2 \frac{1 - N^{-2\beta+1}}{1 - 2^{-2\beta+1}}, \ \beta > 1/2 \qquad \textbf{(15)}$$

where N is the number of wavelet coefficients in the decomposition of the parametric functions as defined in (**3**). These formulas allow us to determine the parameters σ_{TR}^2 and σ_{DEF}^2 if we have an estimate of β and the uncertainty in displacement $\bar{\rho}_{TR}^2$ and $\bar{\rho}_{DEF}^2$.

4. THE FITTING PROBLEM

To solve the fitting problem in bayesian terms we will use the GGSD model to represent the likelihood of deformations around a curve $\bar{\boldsymbol{r}}$ defined through wavelet vector $\bar{\boldsymbol{w}}$. The density function for the mismatch to the data $\boldsymbol{r}_D$ that will be defined as:

$$\mathrm{P}(\boldsymbol{r}_D \mid \boldsymbol{r}) \propto \exp\left(-\frac{1}{2\sigma_D^2} \|\boldsymbol{r} - \boldsymbol{r}_D\|_{L_2(I)}^2 \right) \approx \exp\left(-\frac{1}{2\sigma_D^2} \frac{1}{M} \sum_{i=1}^{M} \left| d\boldsymbol{r}(u_i) - \boldsymbol{F}_W(u_i) d\boldsymbol{w} \right|^2 \right), \qquad \textbf{(16)}$$

$$d\boldsymbol{w} = (\boldsymbol{w} - \bar{\boldsymbol{w}}), \quad d\boldsymbol{r}(u_i) = (\boldsymbol{r}_D - \bar{\boldsymbol{r}})(u_i)$$

that can be written in matrix form as:

$$f_D(d\boldsymbol{w}) \propto \exp\left(-\frac{1}{2\sigma_D^2}\frac{1}{M}(\boldsymbol{D}-\boldsymbol{F}d\boldsymbol{w})^{\mathrm{T}}(\boldsymbol{D}-\boldsymbol{F}d\boldsymbol{w})\right), \boldsymbol{D}=(d\boldsymbol{r}(u_1)^{\mathrm{T}},\cdots,d\boldsymbol{r}(u_M)^{\mathrm{T}})^{\mathrm{T}},\ \boldsymbol{F}=(\boldsymbol{F}_W(u_1)^{\mathrm{T}},\ldots,\boldsymbol{F}_W(u_M)^{\mathrm{T}})^{\mathrm{T}} \quad (\mathbf{17})$$

Then to obtain the maximum a posteriori (MAP) estimator we must solve:

$$\max_{dw} f_B(d\boldsymbol{w})\, f_D(d\boldsymbol{w}) \quad (\mathbf{18})$$

4.1 An approximate MAP solution

In this section we will see how to obtain an approximate solution to the problem in (**18**). To simplify the results, given:

$$\boldsymbol{w}=(c_{0,0;x} \quad d_{0,0;x} \quad \ldots \quad d_{J-1,2^{J-1}-1;x}, c_{0,0;y} \quad d_{0,0;y} \quad \ldots \quad d_{J-1,2^{J-1}-1;y})^{\mathrm{T}}, \quad (\mathbf{19})$$

we define:

$$\tilde{\boldsymbol{w}}_{TR}=(c_{0,0;x} \quad c_{0,0;y})^{\mathrm{T}}, \quad \tilde{\boldsymbol{w}}_{DEF}=(d_{0,0;x} \quad \ldots \quad d_{J-1,2^{J-1}-1;x} \quad d_{0,0;y} \quad \ldots \quad d_{J-1,2^{J-1}-1;y})^{\mathrm{T}}, \quad \tilde{\boldsymbol{w}}=(\tilde{\boldsymbol{w}}_{TR}^{\mathrm{T}},\tilde{\boldsymbol{w}}_{DEF}^{\mathrm{T}})^{\mathrm{T}}. \quad (\mathbf{20})$$

Notice that $\boldsymbol{w}=\boldsymbol{G}\tilde{\boldsymbol{w}}$ where $\boldsymbol{G}$ is a permutation matrix.

We will then use the relation [6]:

$$\|\boldsymbol{d}\|_p^p \approx \sum_i \left(\left(|d_i|^2+\varepsilon\right)^{p/2}-\varepsilon^{p/2}\right), \text{ then we have:} \quad (\mathbf{21})$$

$$\begin{aligned}
&f_B(d\tilde{\boldsymbol{w}}) \propto \exp\left(-\frac{1}{2\sigma_{TR}^2}|d\tilde{\boldsymbol{w}}_{TR}|^2\right)\exp\left(-\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\left(\frac{\eta_2(p)}{\gamma_{jk}}\right)^p\left(\left(\boldsymbol{\Lambda}_{j,k}(d\tilde{\boldsymbol{w}}_{DEF})+\varepsilon\right)^{\frac{p}{2}}-\varepsilon^{\frac{p}{2}}\right)\right)\\
&\gamma_{jk}=\sigma_{DEF}\,2^{-\beta j}\\
&\boldsymbol{\Lambda}_{j,k}(d\tilde{\boldsymbol{w}}_{DEF})=d\tilde{\boldsymbol{w}}_{DEF}^{\mathrm{T}}\boldsymbol{H}_{jk}d\tilde{\boldsymbol{w}}_{DEF}, \quad \boldsymbol{H}_{jk}=\boldsymbol{h}_{j,k,x}\boldsymbol{h}_{j,k,x}^{\mathrm{T}}+\boldsymbol{h}_{j,k;y}\boldsymbol{h}_{j,k;y}^{\mathrm{T}},
\end{aligned} \quad (\mathbf{22})$$

where $\boldsymbol{h}_{j,k,x}$ and $\boldsymbol{h}_{j,k,y}$ are vectors defined as in (**19**) that have a value of 1 in components j,k,x and j,k,y respectively and 0 in the rest.

Taking partial derivatives the optimal solution verifies:

$$\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{D}-\boldsymbol{F}\boldsymbol{G}d\tilde{\boldsymbol{w}})=\boldsymbol{G}\boldsymbol{K}_{d\tilde{w}}d\tilde{\boldsymbol{w}} \quad (\mathbf{23})$$

where $\boldsymbol{K}_{d\tilde{w}}$ is defined as:

$$\begin{pmatrix}
M\sigma_D^2/\sigma_{TR}^2 & 0 & \mathbf{0}\\
0 & M\sigma_D^2/\sigma_{TR}^2 & \mathbf{0}\\
\frac{M\sigma_D^2 p}{2}\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\left(\frac{\eta_2(p)}{\gamma_{jk}}\right)^p\left(\left(\boldsymbol{\Lambda}_{j,k}(\boldsymbol{x}_{DEF})+\varepsilon\right)^{\frac{p}{2}-1}\right)\boldsymbol{H}_{j,k} & &
\end{pmatrix} \quad (\mathbf{24})$$

and the problem can be solved iteratively as:

$$\boldsymbol{F}^{\mathrm{T}}\boldsymbol{D}=(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}+\boldsymbol{G}\boldsymbol{K}_{d\tilde{w}_k}\boldsymbol{G}^{\mathrm{T}})d\boldsymbol{w}_{k+1} \quad (\mathbf{25})$$

5. COMPUTATIONAL RESULTS

In Figure 1 we show a fitting example of a mouse on a laboratory cell. In this case the prior for the contour is the ellipse shown in light grey around the mouse and the final position for the contour is show on black. For each point in the initial contour we look for a corresponding point on the image searching along for the nearest border in the normal direction. We can see that this search will produce a false matching by the presence of the tail of the mouse. When no information about smoothness ($\beta = 0$) is provided this false matching generates a protrusion on the fitting solution as can be seen in the image on the left. However, when we introduce as prior information that the deformation must be smooth ($\beta = 2.5$) the protrusion disappears as can be seen in the image of the right despite the false matching remains. Notice that in contrast with other approaches like snakes, balance of uncertainties between prior and image data measured by their mean square displacement $\overline{\rho}^2$ remains constant in both images, adjusting σ_{DEF}^2 for the change in parameter β.

Figure 1 The fitting problem with different smoothing priors. Values for β are β=0 (left) and β=2.5 (right)

5. CONCLUSIONS

A new model for contour deformations using wavelets has been proposed. It relates contour deformations to Besov smoothness spaces. This allows different degrees of smoothness to be enforced without altering the balance of uncertainty between prior deformation model and data extracted from the image. The fitting problem is solved in bayesian terms and experimental results are shown.

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